Sensitivity of Control-Augmented Structure Obtained by a System Decomposition Method

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The paper presents the verification of a new method for computing behavior sensitivities of a coupled system. The method deals with a system whose analysis can be partitioned into subsets that correspond to disciplines and/or physical subsystems that exchange input-output data with each other. The method uses the partial derivatives of the output with respect to input obtained for each subset separately to assemble a set of linear simultaneous algebraic equations that are solved for the total derivatives of the coupled system response. This sensitivity analysis is verified using an example of a cantilever beam augmented with an active control system to limit the beam's dynamic displacements under an excitation force. The verification shows good agreement of the method with reference data obtained by a finite difference technique involving entire system analysis. The paper also demonstrates the usefulness of a new system sensitivity method in optimization applications by employing a piecewise-linear approach to the same numerical example. The new method's principal merits are its intrinsically superior accuracy in comparison with the finite difference technique, and its compatibility with the traditional division of work in complex engineering tasks among specialty groups.

Nomenclature

A_i = vector of functions
b = beam rectangular cross-sectional width
c = damping constant-design variable vector for the controls subsystem
f = functional relationship
$F(t)$ = dynamic forcing function
G = gain matrix
h = beam rectangular cross-sectional height
I = identity matrix
$J(x,y)$ = Jacobian matrix of $\partial x/\partial y$
M = lumped mass (inertia) matrix
m = mass
N = number of the function vectors A_i
n_i = length of vectors A_i and Y_i
P = Ricatti matrix
r = vector of time-dependent displacements
T_i = static transverse load on beam
u = control input variable
w = vector of static displacements
X = vector of design variables
x = state vector
Y = vector of unknowns in the system analysis
η = generalized coordinate vector
Φ = matrix of natural vibration modes
ρ = user controlled factor in Kreisselmeier-Steinhauser function
ω = natural vibration frequency
ζ_i = critical damping for ith vibration mode

Other symbols used locally are defined where first introduced.

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Introduction

THE merger of structures and active controls in control-augmented structures appears to be one of the most promising means that have recently become available to engineers for making a quantum jump in structural efficiency, especially for dynamic applications in flight and ground vehicles, and in space structures. While extending the design freedom, inclusion of the active controls confronts structures' designers with a coupled system whose behavior is a result of the structures-controls interaction governed by the design variables available in both disciplines. Complexity of that behavior limits the effectiveness of intuition and past experience as design guides, and suggests the use of a formal sensitivity analysis to support human judgment and to provide a basis for numerical optimization.

Recognizing the above, Ref. 1 presents a method for sensitivity analysis of the control-augmented structure, and Ref. 2 reports on the numerical optimization results using the sensitivity data. The sensitivity analysis developed in Ref. 1 is based on the classical, quasianalytical approach of representing the entire system by a single set of governing equations that incorporates terms pertaining to both subsystems—structures and controls. These equations are differentiated with respect to the design variables to yield the sensitivity equations containing the derivatives of behavior as unknowns.

The purpose of the study reported herein is to test a new, alternative method of evaluating the structure-control system behavior derivatives, using the decomposition approach introduced in Ref. 3, and also to demonstrate the use of the method in optimization. In Ref. 3, the sensitivity equations are assembled from building blocks, each block representing the partial sensitivities of each subsystem's output with respect to its input and its design variables. The principal merit of this alternative method is that the partial sensitivity data are obtained from separate subtasks, self-contained within each of the subsystems. This supports the division of large engineering projects among specialty groups, is compatible with the technology of distributed computing, and enables one to use experimental sensitivity data.

Structures-Controls System

It is expedient to use a test case to introduce the method. The test case structure is a simple, cantilever beam, shown in Fig. 1, subjected to static loads and to a dynamic excitation force at the tip. The model is controlled by two sets of actuators—one for the tip lateral displacement and another for the tip rotation. The active controllers limit the beam's dynamic displacements within prescribed bounds. The structure must also be sized to limit the static stresses below allowable levels.

The optimization problem to be solved for this system calls for the minimization of two objectives: the combined structural weight of the beam and the control system, and the control system effort. The control system weight component is assumed to be a simple analytical function of the control effort. Minimization of the control system effort is carried out within the control subsystem by application of the classical linear quadratic controller synthesis.

Constraints are imposed on the static displacements and stresses, natural frequencies, and dynamic displacements. The design variables are the ratio of critical damping to frequency in the control subsystem and the beam cross-sectional dimensions in the structure subsystem. Additional details of the example are available in the Appendix.

In order to perform the optimization, it is necessary to calculate derivatives of the entire system response with respect to the design variables. The new method for sensitivity analysis is implemented for that calculation.

Structures Subsystem

The beam constitutes a structures subsystem in the structure-control system. It is discretized in the span-wise direction into five segments of equal length. In the finite element displacement method used in the study, the model has six nodal points, twelve elastic degrees of freedom (two per nodal point), and five two-dimensional beam finite elements. A lumped mass representation is used to model the beam inertial characteristics.

The structural analysis for the beam dynamic response is comprised of equations for eigenfrequencies and eigenmodes

$$(K - \omega_i{}^2 M)\Phi_i = 0 \tag{1}$$

and of load-deflection equations for the static loads

$$K\, w = T \tag{2}$$

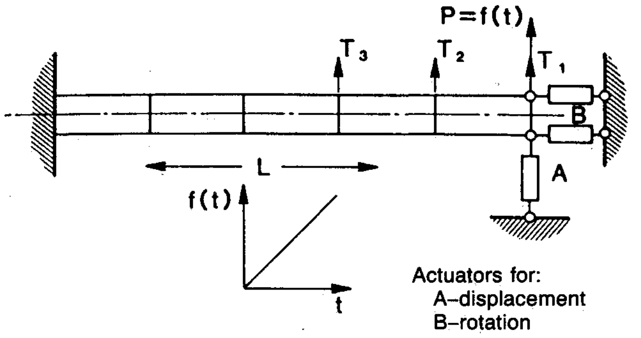

Fig. 1 Cantilever beam with actuators.

where w is a vector of static displacements and T is the load vector. Two different sets of design variables are used in the structure subsystem. The first set includes the reciprocal of the cross-sectional bending moment of inertia and the cross-sectional area of each segment of the beam. The other combination is the width and height of the rectangular beam cross section for each segment.

Controls Subsystem

It is assumed that the actuators are part of an optimal control system with a linear quadratic regulator, so that the control system is described by the following equations. The equations of motion for the dynamical system are

$$[M]\{\ddot{r}\} + [C]\{\dot{r}\} + [K]\{r\} = [b]\{u\} + [\bar{b}]\{F\} \tag{3}$$

A modal transformation, that can be written as

$$\{r\} = [\Phi]\{\eta\} \tag{4}$$

enables one to define a state vector as

$$\{x\} = \begin{Bmatrix} \eta \\ \dot{\eta} \end{Bmatrix} \tag{5}$$

in terms of which the state equation becomes

$$\{\dot{x}\} = [A]\{x\} + [B]\{u\} + [\bar{B}]\{F\} \tag{6}$$

where

$$[A] = \begin{bmatrix} 0 & I \\ -\omega^2 & -2\zeta\omega \end{bmatrix}$$

$$[B] = \begin{bmatrix} 0 \\ \Phi^T b \end{bmatrix} \qquad [\bar{B}] = \begin{bmatrix} 0 \\ \Phi^T \bar{b} \end{bmatrix} \tag{7}$$

The optimal control problem is one in which the following quadratic performance index is minimized

$$J = \int_0^\infty (\{x\}^T[Q]\{x\} + \{u\}^T[R]\{u\})\, \mathrm{d}t \tag{8}$$

subject to

$$\{\dot{x}\} = [A]\{x\} + [B]\{u\}$$

where $[Q]$ and $[R]$ are arbitrary weighting matrices defined in terms of prescribed constants $\bar{Q}$ and $\bar{R}$ as

$$[Q] = \bar{Q}[I] \qquad \text{and} \qquad [R] = \bar{R}[I]$$

The ratio of these constants determines the amount of control effort permitted. The solution of the above linear quadratic regulator problem results in the nonlinear algebraic Riccati equation

$$[A]^T[P] - [P][B][R]^{-1}[B]^T[P] + [P][A] + [Q] = 0 \tag{9}$$

The solution of this equation yields the symmetric positive definite matrix referred to as the Riccati matrix, which is then used to obtain the gain matrix from the relationship

$$[G] = [R]^{-1}[B]^T[P] \tag{10}$$

The control input vector, computed from the system response using the gain matrix, can now be written as

$$\{u\} = -[G]\{x\} \tag{11}$$

and substituted into the state equation so that the controlled system response may be obtained from

$$\{\dot{x}\} = [\bar{A}]\{x\} + [\bar{B}]\{F\} \tag{12}$$

where

$$[\bar{A}] = [A] - [B][G] \tag{13}$$

Once the state vector has been found from integration of Eq. (12), by a time-stepping method using a finite series algorithm, the dynamic displacement vector may be obtained from Eq. (4).

The design variable in the control system is the proportionality constant c, which defines the damping factor ζ_i, for the ith eigenmode, as follows:

$$\zeta_i = c\ \omega_i/2 \tag{14}$$

In the above analysis, this variable affects the state solution via the matrix $[A]$.

System Sensitivity Analysis

Solution to the structure-control system is based on the algorithm introduced in Ref. 3. The essence of the referenced approach is stated first, followed by its adaptation to the problem at hand.

Generic Solution

Taking the approach of Ref. 3, the system's governing equations are partitioned, with each partition corresponding to a distinct engineering discipline and/or a physically distinct subsystem. The following notation defines the ith partition

$$A_i((X, \ldots, Y_j, \ldots), Y_i) = 0 \tag{15}$$

where A_i is a vector of functions equated to zero, so that they form a set of n_i equations. The quantities in the inner parentheses are the inputs that must be given in order to solve Eq. (15) for Y_i. These inputs entail the independent variables X, which include design variables and externally prescribed constants, and dependent variables Y_j, $j \neq i$. The latter are obtained from solutions of the other partitions A_j, $j \neq i$, and represent couplings of A_i to A_j. The couplings require that the entire set of governing equations composed of the partitions, such as Eq. (15), must be solved as a set of simultaneous equations. An iterative solution method is commonly used for large, complex, and nonlinear systems.

When the solution Y, composed of all the partitions Y_i, is obtained for a given vector X, the sensitivity derivatives of Y_i with respect to X are computed as a solution of a set of simultaneous, linear, algebraic equations

$$\begin{bmatrix} I & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & I & \cdot\ \cdot & -J_{ij} \\ & & \cdot\ \cdot & \cdot\ \cdot & \cdot \\ & & \cdot\ \cdot & \cdot\ \cdot & \cdot \\ & & -J_{ji} & \cdot\ \cdot & I \end{bmatrix} \begin{Bmatrix} \cdot \\ \cdot \\ \cdot \\ \dfrac{dY_i}{dX_k} \\ \cdot \\ \cdot \\ \dfrac{dY_j}{dX_k} \end{Bmatrix} = \begin{Bmatrix} \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial f_i}{\partial X_k} \\ \cdot \\ \cdot \\ \dfrac{\partial f_j}{\partial X_k} \end{Bmatrix} \tag{16}$$

In these equations, the matrix of coefficients has unities on the diagonal and its off-diagonal submatrices are the negatives of the Jacobian matrices of the partial derivatives of output, with respect to input for the coupled A_i. Specifically, expressing Y_i as an implicit function of the inputs to A_i

$$Y_i = f_i(X, \ldots, Y_j, \ldots), \qquad j \neq i \tag{17}$$

the Jacobian J_{ij} is

$$J_{ij} = \left[\frac{\partial f_i}{\partial Y_j}\right], \qquad n_i \times n_j \tag{18}$$

and is determined by finite difference. It consists of the partial sensitivity information, obtainable locally for each Y_i by treating it as a function of given arguments, according to Eq. (17).

The right-hand side vector in Eq. (16) contains the partial derivatives of Y_i, with respect to a particular X_k, one at a time; also computable locally from Eq. (17).

Application to Numerical Example

Applying the approach described in Ref. 3 to the example case, we define the structure-control system by a schematic representation, shown in Fig. 2, where S represents the structures subsystem and C represents the controls subsystem. The vector Y includes interaction of one subsystem with the other. The physical influence of the controls subsystem on the structures subsystem is through the actuator forces. However, the system equations of motion [Eq. (3)] and evaluation of the dynamic displacement constraints have been included in the analysis of the controls subsystem. Consequently, in the partitioned mathematical model of the system, the controls-to-structures data channel needs to transmit only the control system mass data that has to be accounted for in the vibration analysis in order to compute natural vibration frequencies and modes.

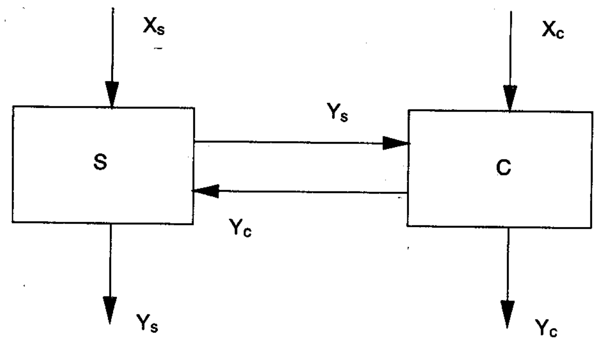

Fig. 2 Coupled structure-control system.

Table 1 Structures—controls coupling data and design variables

	Structures subsystem (Eqs. 1–2)
Y_S	• output from structures, input to controls
	Φ – matrix of natural vibration modes
	ω – array of the natural vibration frequencies
	• output from structures to outside
	W_S – structures weight
	g_1–g_5 – values of constraints on static displacements, natural frequencies, and static stresses
X_S	• design variables directly influencing structure
	w,h – width and height of the beam rectangular cross section
	Controls subsystem (Eqs. 3–13)
Y_C	• output from controls, input to structures
	m_C – control subsystem mass participating in the beam vibration
	• output from controls to outside
	W_C – control subsystem weight
	g_6,g_7 – values of dynamic displacement constraints
X_C	• design variables directly influencing controls
	c – ratio of critical damping factor to frequency

The other data channel represents the structure's influence on the control system through the vibration frequency and mode shapes. In addition to the data coupling the two subsystems, there is data output to the outside representing response of the entire system. This data is the output from each subsystem, as follows:

1) Structures: weight and the values of the static stress and displacement constraints

2) Controls: weights and the values of the dynamic displacement constraints

Table 1 summarizes the coupling data and the design variables for both subsystems, and additional numerical details are given in the Appendix.

The generic governing equations may be written, in this particular case, as partitioned into the structural analysis S and control analysis C, and coupled by the presence of the output of one in the input of the other.

$$S([X_s, Y_c], Y_s) = 0 \tag{19}$$

$$C([X_c, Y_s], Y_c) = 0 \tag{20}$$

These equations correspond to the groups of Eqs. (1-2) and (3-13), respectively.

When taken separately, Eqs. (19) and (20) yield the interactions Y as implicit functions of the design variables, X.

$$Y_s = f_s(X_s, Y_c) \text{ and } Y_c = f_c(X_c, Y_s) \tag{21}$$

When solved simultaneously, Eqs. (19) and (20) produce the Y unknowns as functions of the variables X_s and X_c.

Accounting for Eq. (21), the generic sensitivities of Eq. (16) take on the following form:

$$\begin{bmatrix} I & -J(f_s, Y_c) \\ -J(f_c, Y_s) & I \end{bmatrix} \begin{Bmatrix} \dfrac{dY_s}{dX_*} \\ \dfrac{dY_c}{dX_*} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial f_s}{X_s} \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ \dfrac{\partial f_c}{\partial X_c} \end{Bmatrix} \tag{22}$$

where the character '*' may be substituted by either s or c. The sensitivity solution, dY/dX, is obtained for one design variable at a time.

The partial derivative data that make up Eq. (22) are obtainable separately within each of the participating "black boxes" of structures and control by any technique available, e.g., analytically, by finite differences, or they could even be produced experimentally. A simple, one-step-forward finite difference technique is used in this study in both the structures and controls subsystems.

Verification of the System Sensitivity Analysis and Results

To verify the system sensitivity method represented by Eq. (22), the following stepwise procedure was implemented.

1) The structure-control system is analyzed by solving Eqs. (19) and (20) for Y for initial values of X.

2) Partial derivatives with respect to X and Y are computed by finite difference for structures and controls separately, using, as reference values, the X and Y satisfying Eqs. (19) and (20).

3) The partial derivatives obtained above are input to the system sensitivity [Eq. (22)], which are then solved for the total derivatives with respect to the X.

4) The system sensitivities of Y with respect to X are calculated again by a finite difference technique, this time involving the coupled system analysis, i.e., Eqs. (19) and (20), and using the same Y and X reference values as in step 2.

5) The system sensitivities obtained in steps 3 and 4 are compared.

This comparison is illustrated by typical data samples in Tables 2, 3, and 4 that display sensitivities obtained for the following data:

Beam length	500	in.
Beam moment of inertia	144	$in.^4$
Beam cross-sectional area	48	$in.^2$
Damping constant, c	0.01	
Material properties for Al-alloy		

The constant parameters (gain levels) are set to achieve very low (almost negligible), moderate, and high levels of the control presence in the structural dynamics by changing the weighting matrices in the controls performance index. The constants associated with the weighting matrices for the three cases are as follows:

	Low	Moderate	High
$\bar{Q}$	1.0	1.0	1.0
$\bar{R}$	0.0001	0.01	1.0

The results are shown in Tables 2, 3, and 4, respectively.

All three tables pertain to the controls design variable and the columns in each table are, from the left: results from step 4, results from step 3, and relative error of the results from step 3 with respect to the results from step 4.

Table 2 Comparison of finite difference and GSE sensitivities for low-control level

Finite difference	GSE	% difference
	DYs/DXc	
0.63960E+01	0.63887E+01	0.11
0.20304E+03	0.20184E+03	0.59
0.58682E−02	0.85049E−02	0.74
0.26325E−01	0.26024E−01	1.14
0.15871E−03	0.15833E−03	0.24
0.32208E−03	0.32118E−03	0.28
0.29355E−01	0.29358E−01	0.01
0.31789E−01	0.31317E−01	1.49
0.24835E−03	0.24771E−03	0.26
−.31898E−03	−.32132E−03	0.73
0.56326E−01	0.56031E−01	0.52
−.42717E−01	−.42761E−01	0.10
0.27978E−03	0.27713E−03	0.95
−.11067E−02	−.10984E−02	0.76
0.84043E−01	0.83459E−01	0.69
−.16504E+00	−.16436E+00	0.41
0.26853E−03	0.26734E−03	0.45
−.11968E−02	−.11940E−02	0.23
0.11007E+00	0.10937E+00	0.64
−.26683E+00	−.26588E+00	0.36
0.25456E−03	0.25362E−03	0.37
−.88010E−03	−.87980E−03	0.03
0.00000E+00	0.00000E+00	0.00
0.00000E+00	0.00000E+00	0.00
0.00000E+00	0.00000E+00	0.00
−.31988E−01	−.31760E−01	0.71
−.12755E+00	−.12707E+00	0.38
0.00000E+00	0.00000E+00	0.00
	DYc/DXc	
−.54166E+00	−.54149E+00	0.03
−.20907E+03	−.20905E+03	0.01
0.88215E−01	0.89733E−01	1.69
0.13431E+00	0.13612E+00	1.33

Table 3 Comparison of finite difference and GSE sensitivities for moderate control level

Finite difference	GSE	% difference
	DYs/DXc	
0.99818E+00	0.10989E+00	9.17
0.34587E+02	0.38611E+00	10.42
0.62088E−03	0.14330E−02	56.67
−.56674E−01	0.50202E−02	108.86
0.82849E−04	0.26614E−04	67.88
0.51425E−02	0.62056E−04	98.79
0.46690E−02	0.49235E−02	5.17
0.31988E−01	0.60925E−02	80.95
−.38805E−06	0.41292E−04	100.94
−.33497E−02	−.60622E−04	98.19
0.83447E−02	0.93419E−02	10.67
−.24339E−01	−.78102E−02	67.91
0.69073E−04	0.45602E−04	33.98
0.21389E−02	−.20424E−03	109.55
0.12318E−01	0.13824E−01	10.89
−.23010E−01	−.29951E−01	23.18
0.15522E−04	0.43346E−04	64.19
−.20629E−02	−.21153E−03	89.75
0.15895E−01	0.17997E−01	11.68
−.19669E−01	−.47254E−01	58.38
0.51999E−04	0.40773E−04	21.59
0.12511E−02	−.14467E−03	111.56
0.00000E+00	0.00000E+00	0.00
0.00000E+00	0.00000E+00	0.00
0.00000E+00	0.00000E+00	0.00
−.49671E−02	−.52758E−02	5.80
−.21060E−01	−.23602E−01	10.77
0.00000E+00	0.00000E+00	0.00
	DYc/DXc	
−.77337E−01	−.81238E−01	4.80
−.29844E+02	−.31440E+02	5.08
0.66012E−01	0.87562E−01	24.61
0.79026E−01	0.11993E+00	34.11

Tabel 4 Comparison of finite difference and GSE sensitivities for high-control level

Finite difference	GSE	% difference
	DYs/DXc	
0.78328E+01	0.78276E+01	0.07
0.29663E+03	0.29595E+03	0.23
0.10070E−01	0.10081E−01	0.11
0.38991E−01	0.38653E−01	0.87
0.18704E−03	0.18706E−03	0.01
0.44703E−03	0.47787E−03	6.45
0.34322E−01	0.34486E−01	0.48
0.46889E−01	0.47030E−01	0.30
0.28638E−03	0.28819E−03	0.63
−.43656E−03	−.46235E−03	5.58
0.64969E−01	0.65184E−01	0.33
−.58313E−01	−.58194E−01	0.20
0.31665E−03	0.31516E−03	0.47
−.15514E−02	−.15342E−02	1.11
0.96162E−01	0.96032E−01	0.13
−.22199E+00	−.22184E+00	0.07
0.29802E−03	0.29639E−03	0.55
−.15196E−02	−.15293E−02	0.64
0.12438E+00	0.12448E+00	0.08
−.34272E+00	−.34279E+00	0.02
0.28095E−03	0.27663E−03	1.54
−.98255E−03	−.97852E−03	0.41
0.00000E+00	0.00000E+00	0.00
0.00000E+00	0.00000E+00	0.00
0.00000E+00	0.00000E+00	0.00
−.36558E−01	−.36709E−01	0.41
−.17767E+00	−.17709E+00	0.33
0.00000E+00	0.00000E+00	0.00
	DYc/DXc	
−.52992E+00	−.52977E+00	0.03
−.20453E+03	−.20448E+03	0.03
−.84192E−01	−.78970E−01	6.20
−.16570E+00	−.16844E+00	1.63

The sample shows that the agreement of the results produced by the method proposed in Ref. 3, represented herein by Eq. (22), agree very well with the finite difference verification. The only discrepancies correspond to the relatively small derivative values, and are attributed not to error in the method being tested, but to a loss of accuracy of the finite difference technique caused by the limited word length in the computer. The sample shows also that the errors relative to the finite difference results are greater for the case of moderate level of control (Table 3). This is so because the effect of the small finite difference step gets submerged in the numerical error of the simultaneous solution of Eqs. (19) and (20)—a confirmation of the effect predicted in Ref. 3. The comparisons for other design variables of the problem were found to be qualitatively the same.

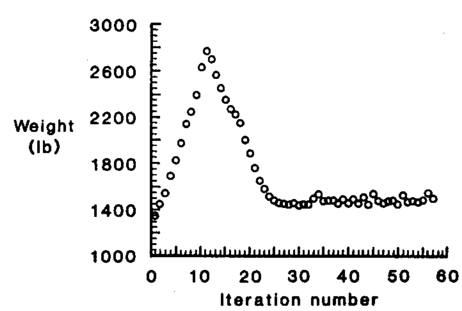

Fig. 3 Histogram of optimization started from a feasible design.

System Sensitivity Derivatives Used in Optimization

System sensitivities quantify "what if" questions that are an intrinsic part of the design process. They can also be used to guide a formal optimization algorithm. To demonstrate their usefulness in the latter, the structure-controls system was optimized by a piecewise-linear optimization method, using the system sensitivity derivatives calculated via Eq. (22) at each linear stage.

The mathematical statement of the optimization problem is given in the Appendix, and the optimization procedure follows a sequence in which the system is initialized and a solution of this initial system [Eqs. (19) and (20)] is obtained. The partial derivatives for the structure and control subsystems are computed separately (see step 2 in the procedure listed in the previous section), followed by a solution of Eq. (22) for derivatives of the system response with respect to the design variables. The system is then optimized by approximating the objective function and constraints by linear extrapolation with respect to design variables, using the derivatives obtained above (see the Appendix for details of the formulation), and restricting the change of design variables by prescribing lower and upper bounds. The results from this design are used as the new set of variables for which the sensitivities must be obtained. This procedure is repeated until constraints are satisfied and there is no appreciable change in the objective function for three cycles of design.

The optimization procedure performed as expected. Figures 3 and 4 illustrate typical histograms of the objective function of weight vs iteration number for a feasible and infeasible initial design point, respectively. The usable-feasible direc-

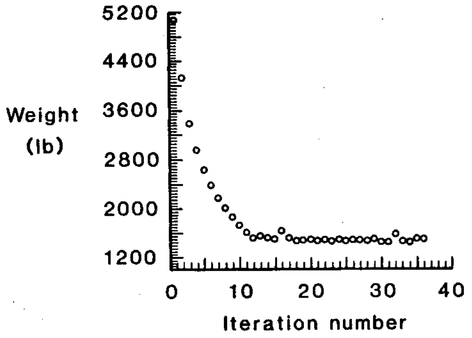

Fig. 4 Histogram of optimization started from an infeasible design.

tions algorithm 4 is used for searching the design space in step 5. The overshoot peak of the histogram for optimization started from an infeasible point (Fig. 4) and is typical for the usable-feasible directions algorithm. Although the problem is highly nonlinear, the procedure converged to essentially the same final design (see Appendix), confirming the system sensitivity analysis formulated in Ref. 3 as useful in optimization.

Conclusions

A method proposed in Ref. 3 for computing sensitivities of an internally coupled system to design variables was tested by application to a simple, control-augmented structure. The method yields the system sensitivity from the solution of linear, algebraic equations built from the partial sensitivity information obtained independently for each of the subsystems.

Comparison of the results with the reference results produced by a finite difference technique applied to the solution of the entire system confirmed good performance of the method of Ref. 3. Also, it demonstrated that the finite difference technique, when applied to an internally coupled system, may lose accuracy, due to the system solution noise overwhelming the effect of the small finite difference step.

The usefulness of the sensitivities obtained in a manner presented in the paper was demonstrated by a successful application in the formal optimization of a control-augmented structure. The principal benefits of the system sensitivity analysis implemented in this paper are an improved accuracy in comparison to the finite difference method involving the entire system analysis, and further, the decomposition of the system sensitivity analysis into partial sensitivity analyses performed separately in each discipline and subsystem. This should support a broad workfront of people and computers in design organization and allow use of specialized sensitivity analysis methods unique to each discipline.

Appendix: Details of Numerical Examples

Definition of the Example

The optimization problem is stated

$$\text{minimize} \quad W_t(X) = W_s(X) + W_c(X) \tag{A1}$$

subject to

$$g(X) = f(\text{static tip displacements})$$
$$g(X) = f(\text{natural frequencies})$$
$$g(X) = f(\text{static stresses})$$
$$g(X) = f(\text{dynamic displacements}) \tag{A2}$$

$$X_L \le X \le X_U \tag{A3}$$

The following extrapolations are made in approximate analysis, coupled to an optimization program:

$$W_t^{i+1} = W_t^{i} + \frac{d(W_s + W_c)^i}{dX_s}\Delta X_s + \frac{d(W_s + W_c)^i}{dX_c}\Delta X_c \tag{A4}$$

$$g_k^{i+1} = g_k^{i} + \frac{dg_k^{i}}{dX_s}\Delta X_s + \frac{dg_k^{i}}{dX_c}\Delta X_c \tag{A5}$$

Move limits ranging from 2.5% to 10% are included in (A3). The constraints represented in Eq. (A2) are as follows:

$$g_1 = \frac{dl}{dl_a} - 1 \le 0 \qquad g_5 = \frac{\sigma}{\sigma_a} - 1 \le 0$$
$$g_2 = \frac{dr}{dr_a} - 1 \le 0 \qquad g_6 = \frac{ddl}{ddl_a} - 1 \le 0$$
$$g_3 = -\frac{\omega_1}{\omega_{1_a}} + 1 \le 0 \qquad g_7 = \frac{ddr}{ddr_a} - 1 \le 0$$
$$g_4 = -\frac{\omega_2}{\omega_{2_a}} + 1 \le 0 \tag{A6}$$

where the prescribed limits are

Symbol	Description	Allowable limit	
dl	static lateral displacement	50.00	in.
dr	static rotational displacement	0.20	rad
ω_1	first natural frequency	1.00	Hz
ω_2	second natural frequency	1.25	Hz
ddl	dynamic lateral displacement	50.00	in.
ddr	dynamic rotational displacement	0.15	rad
σ	static stress	30,000	psi (A7)

The stress constraint g_5 is formulated as a single cumulative constraint representing static stress in all the elements of the beam, due to static loads T_1, T_2, T_3. The cumulative constraints taken in form of the Kreisselmeier-Steinhauser function (KS function) introduced in Ref. 5. The KS is a continuous, differentiable envelope of a family of functions, so that for constraint functions g_j, where L is the number of constraints being represented in the function

$$KS(g_j) = (1/\rho)\, ln\left(\sum_{j=1}^{L} \exp(\rho\, g_j)\right) \tag{A8}$$

The KS function has the property of

$$\max(g_j) \le KS(g_j) \le [\max(g_j) + ln(L)/\rho] \tag{A9}$$

The effectiveness of this function in structural optimization has been reported many times (for example, see Ref. 6). In this application, the KS-based cumulative constraints are also used in (A6) for constraints g_6 and g_7 to represent dynamic displacements due to dynamic force $F(t)$ at several discrete time intervals.

The design variable and behavior variable vector for the structure subsystem are

$$X_s = ([b,h]_1, [b,h]_2, \ldots, [b,h]_5)^T \tag{A10}$$

$$Y_s = (\omega_1^2, \omega_2^2, \omega_3^2, \Phi_7, \ldots, \Phi_{36}, W_s, g_1, \ldots, g_5)^T \tag{A11}$$

Here, $\Phi_7, \ldots, \Phi_{36}$ denote the nonzero components of the first three eigenmodes of the system.

The design variable and behavior variable vector for the control subsystem are

$$X_c = (c) = \frac{2\,\zeta_i}{\omega_i} \tag{A12}$$

$$Y_c = (m_c, W_c, g_6, g_7)^T \tag{A13}$$

where the control subsystem weight is approximated as a function of the control effort by an analytical expression

$$W_c = \sum_{i=1}^{q} K_c \mid u_i \mid^{1/2} \tag{A14}$$

Table A1 Comparison of initial and final designs

Case	W_t	X_{s1}	X_{s2}	X_{s3}	X_{s4}	X_{s5}	X_{s6}	X_{s7}	X_{s8}	X_{s9}	X_{s10}	X_{c1}
Feasible initial design												
Initial	5064.17	10.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0	0.01
Final	1509.17	3.152	14.18	3.113	12.39	3.000	10.14	3.000	6.958	3.000	3.372	0.06
Infeasible initial design												
Initial	1347.48	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	0.01
Final	1502.48	3.148	14.25	3.012	12.45	3.043	10.12	3.000	6.957	3.000	3.198	0.06

where q is the number of control inputs and K_c is a prescribed constant, taken as 2.0 in this example.

The numerical data for the example are

$$
\begin{aligned}
E &= 10.5 * 10^6 \text{ psi} \\
\rho_s &= 0.1 \text{ slug/in.}^3 \\
\nu &= 0.3 \\
L &= 500 \text{ in.} \\
T_1 &= 1000 \text{ lb} \\
T_2 &= 5000 \text{ lb} \\
T_3 &= 1000 \text{ lb}
\end{aligned}
\tag{A15}
$$

The excitation force is given as a ramp function of time, such that

$$F(t) = k*t \tag{A16}$$

where $k = 1000$ lb/s and the time t varies from 0.0 to 2.0 s. The entire optimization problem is limited to the two second time period for the dynamic response.

Optimization Results

The optimization results for two cases (initially feasible design and initially infeasible design) are displayed in Table A1, in addition to the initial design values. The lower and upper bounds for the design variables for both cases are

$$(X_{sL}, X_{sU}) = (3.0, 36.0) \text{ in.}$$

and

$$(X_{cL}, X_{cU}) = (0.01, 0.06)$$

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References

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